Note Title

C2 Exercise 7D (sum of a geometric series)

27/04/2013

It's not one of the questions, but the proof of the standard result is a real joy, so I'll share it here first:

A geometric <u>sequence</u> is $a, ar, ar^2, ar^3, \ldots, ar^{n-1}, ar^n, \ldots$

The corresponding <u>series</u> is these terms added up.

The sum of the first n terms is called the 'sum to n' and written S_n.

So
$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}.$$

We want a quick way to work this out without having to find all the terms and add them up.

The 'lightbulb moment' is to multipy S_n by the ratio r.

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n.$$

Notice <u>every</u> term is multiplied by r.

Now notice how almost all the terms on the RHS of S_n are the same still as those on the RHS of rS_n. So they cancel:

$$S_n - rS_n = a + \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{\ldots} + \cancel{ar^{n-2}} + \cancel{ar^{n-1}} - (\cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \cancel{ar^4} + \cancel{\ldots} + \cancel{ar^{n-1}} + ar^n)$$

$$\Rightarrow S_n - rS_n = a - ar^n$$

now factorise the 'S_n' out on the LHS and the 'a' on RHS

$$\Rightarrow S_n(1-r) = a(1-r^n)$$

$$\Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

1 Find the sum of the following geometric series (to 3 d.p. if necessary)

a $1 + 2 + 4 + 8 + \ldots$ (8 terms)

$a = 1 \quad r = 2 \quad n = 8$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_8 = \frac{1(1-2^8)}{1-2} = 2^8 - 1 = 255$$

keep stating the general rule... it will help you memorize it.

b $32 + 16 + 8 + \ldots$ (10 terms)

$a = 32, \ r = \frac{1}{2}, \ n = 10$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_{10} = \frac{32(1-(\frac{1}{2})^{10})}{1-\frac{1}{2}} = 63.9375$$

c $4 - 12 + 36 - 108 + \ldots$ (6 terms)

$a = 4, \ r = -3, \ n = 6$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_6 = \frac{4(1-(-3)^6)}{1--3} = \frac{4(1-729)}{4} = -728$$

d $729 - 243 + 81 - \ldots - \frac{1}{3}$ (you work out how many terms)

$a = 729, \ r = -\frac{1}{3}$

$$u_n = ar^{n-1} \Rightarrow -\frac{1}{3} = 729 \times \left(-\frac{1}{3}\right)^{n-1} \Rightarrow -\frac{1}{2187} = \left(-\frac{1}{3}\right)^{n-1}$$

$$\Rightarrow 3^{n-1} = 2187 \Rightarrow n-1 = \frac{\log_{10} 2187}{\log_{10} 3} \Rightarrow n-1 = 7 \Rightarrow n = 8$$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_8 = \frac{729(1-(-\frac{1}{3})^8)}{1--\frac{1}{3}} = \frac{1640}{3} \text{ or } 546\tfrac{2}{3}.$$

e $\sum_{r=1}^{6} 4^r \Rightarrow a = 4^1 = 4$

common ratio is also 4.

we're adding from 1 to 6 so there are 6 terms: $n = 6$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow \sum_{r=1}^{6} 4^r = \frac{4(1-4^6)}{1-4} = 5460$$

f $\quad \sum_{r=1}^{8} 2\times(3)^r$

first term $a = 2\times(3)^1 = 6$

common ratio $r=3$

there are $n=8$ terms

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow \sum_{r=1}^{8} 2\times(3)^r = \frac{6(1-3^8)}{1-3} = 19680$$

1g $\quad \sum_{r=1}^{10} 6\times\left(\frac{1}{2}\right)^r$

first term $a = 6\times\left(\frac{1}{2}\right)^1 = 3$

common ratio $r=\frac{1}{2}$

there are $n=10$ terms

so $$\sum_{r=1}^{10} 6\times\left(\frac{1}{2}\right)^r = \frac{3\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}} = \frac{3069}{512} = 5.994140625$$

h $\quad \sum_{r=0}^{5} 60\times\left(\frac{-1}{3}\right)^r$

first term is $a = 60\times\left(\frac{-1}{3}\right)^0 = 60$

$r = -\frac{1}{3}$

there are $n=\underline{6}$ terms since we're counting from zero!

$$\sum_{r=0}^{5} 60\times\left(\frac{-1}{3}\right)^r = \frac{60\left(1-\left(-\frac{1}{3}\right)^6\right)}{1-\left(-\frac{1}{3}\right)} = \frac{3640}{81} \approx 44.94$$

to 2 d.p.

2 The sum of the first three terms of a geometric series is 30.5. If the first term is 8, find possible values of r.

$$S_3 = \frac{8(1-r^3)}{1-r} = 30.5$$

$$\Rightarrow \quad 8 - 8r^3 = \frac{61}{2} - \frac{61}{2}r$$

$$\Rightarrow \quad 16 - 16r^3 = 61 - 61r$$

$$\Rightarrow \quad 16r^3 - 61r + 45 = 0$$

$r = 1 \Rightarrow f(r) = 0$ so $(r-1)$ is a factor

$$\Rightarrow \quad (r-1)(16r^2 + 16r - 45) = 0$$

$$\Rightarrow \quad (r-1)(4r+9)(4r-5) = 0$$ (Not an easy factorisation)

$$\Rightarrow \quad r = -\frac{9}{4} \text{ or } r = \frac{5}{4}.$$

$r=1$ only <u>appears</u> to work because then we have a division by zero in $S_n = \frac{a(1-r^n)}{1-r}$.

3 The man who invented the game of chess was asked to name his reward. He asked for 1 grain of corn to be placed on the first square of his chessboard, 2 on the second, 4 on the third until all 64 squares were covered. He then said he would like as many grains of corn as the chessboard carried. How many grains of corn did he claim as his prize? see question 1a) above.

$$1 + 2 + 4 + \dots + 2^{63}$$

$a = 1$ $\quad n = 64$

$r = 2$

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow S_{64} = \frac{1(1-2^{64})}{1-2}$$

$$S_{64} = 2^{64} - 1$$

$$\simeq 1.84 \times 10^{19}$$

actually $S_{64} = 18446744073709551615$

4 Jane invests £4000 at the start of every year. She negotiates a rate of interest of 4% per annum, paid at the end of every year. How much is her investment worth at the end of the 10^{th} and the 20^{th} years?

Tabulate. This is harder than it first appears as the first term of the series is the last £4000 she invested and the last term of the series is the £4000 she invested in the first year.

for each £4000 invested by the end of the year it is worth

end year 1 $\quad u_1 = 4000 \times 1.04 = £4160.00$

end year 2 $\quad u_2 = 4000 \times 1.04^2 = £4326.40$

end year 3 $\quad u_3 = 4000 \times 1.04^3 \simeq £4499.46$

end year n $\quad u_n = 4000 \times 1.04^n$

The total value of the investment at the end of the n^{th} year is the sum of n such amounts above, with the smallest amount being the most recent investment.

So $\quad S_1 = 4160 = £4160$ (after 1 year Jane's got interest on only her first £4000)

$S_2 = 4160 + 4326.40 = £8486.40$

(but in the second year the first £4000 has grown again and the second £4000 has started to earn money too.)

$$S_3 \simeq \underbrace{4160}_{\text{money invested at start of year 3}} + \underbrace{4326.40}_{\text{money invested at start of year 2}} + \underbrace{4499.46}_{\text{money invested at start of year 1}} = £12\,985.86$$

$$S_n = \sum_{k=1}^{n} 4000 \times 1.04^k$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

here $a = 4160$ NOT £4000.

$r = 1.04$

$n = 10$

$$S_{10} = \frac{4160(1-1.04^{10})}{1-1.04} = 49945.40563$$

so after 10 years her investment is worth £49945.41

and

$$S_{20} = \frac{4160(1-1.04^{20})}{1-1.04} = 123876.8069$$

so after 20 years her investment is worth £123876.81.

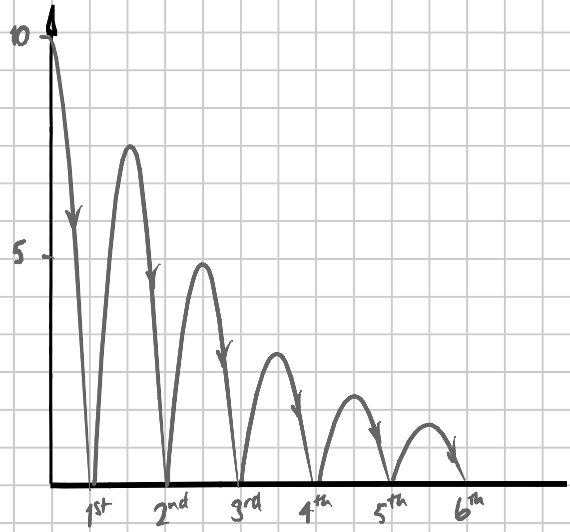
It's worth noting that even in the low interest-rate climate of 2013, 4% interest is not inconceivable in a fixed-term bond and saving £330 per month is just about possible.

There are some strong financial-literacy messages here.

5 A ball is dropped from a height of 10m. It bounces to a height of 7m and continues to bounce. Subsequent heights to which it bounces follow a geometric sequence.

a) Find out how high it will bounce after the fourth bounce and

b) the total distance travelled from release until it hits the ground for the sixth time.



define u_n to be the maximum height reached after the n^{th} bounce. Then

$$u_1 = 7$$

$$u_2 = 7 \times \frac{7}{10} = 4.9$$

$$u_3 = 7 \times \left(\frac{7}{10}\right)^2 = 3.43$$

$$u_4 = 7 \times \left(\frac{7}{10}\right)^3 = 2.401$$

a) So after the fourth bounce the maximum height reached is 2.40m.

b) total distance covered $= 10 + 2\sum_{k=1}^{5} u_k$

(Since it falls 10m before the first bounce and thereafter each height is doubled to account for the distance travelled on the way up and the way back down)

$$\sum_{k=1}^{5} u_k = S_5 = \frac{a(1-r^n)}{1-r} \quad \text{where } a=7,\ r=0.7,\ n=5$$

$$\Rightarrow \sum_{k=1}^{5} u_k = \frac{7(1-0.7^5)}{1-0.7} = 19.4117$$

So the total distance $= 10 + 2 \times 19.4 = 48.8234$m

6 Find the least value of n such that the sum $3 + 6 + 12 + 24 + \ldots$ to n terms would first exceed 1.5 million.

$a = 3 \qquad r = 2 \qquad n$ is unknown

put $S_n = 1500000$ and then round up n to next integer

$$S_n = \frac{a(1-r^n)}{1-r} \Rightarrow 1500000 = \frac{3(1-2^n)}{1-2}$$

$$\Rightarrow -1500000 = 3 - 3 \times 2^n$$

$$\Rightarrow 3 \times 2^n = 1500003$$

$$\Rightarrow 2^n = 500001$$

$$\Rightarrow \log_{10} 2^n = \log_{10} 500001$$

$$\Rightarrow n = \frac{\log_{10} 500001}{\log 2}$$

$$n = 18.93157143$$

So to <u>exceed</u> 1.5 million we need 19 terms.

7 Find the least value of n such that the sum 5 + 4.5 + 4.05 + ... to n terms would first exceed 45.

$a = 5$ $r = 0.9$ n is unknown to be found

put $S_n = 45$ and round up n to next integer

$$\frac{5(1-0.9^n)}{1-0.9} = 45$$

$$1 - 0.9^n = 0.9$$

$$0.9^n = 0.1$$

$$\Rightarrow \quad n = \frac{\log_{10} 0.1}{\log_{10} 0.9}$$

$$n = 21.85434533$$

so we need 22 terms.

8 Richard is sponsored to ride 1000 miles over a number of days. He cycles 10 miles on day 1 and increases this by 10% per day. How long will it take him to complete the challenge and (if he stops at 1000 miles) what is the greatest distance he cycles on any single day?

$u_1 = 10$ $\qquad$ $a = 10$

$u_2 = 11$ $\qquad$ $r = 1.1$

$u_3 = 12.1$ $\qquad$ n is unknown and to be found

$u_n = 10 \times (1.1)^{n-1}$ $\qquad$ put $S_n = 1000$ and round n up.

$$1000 = \frac{10(1 - 1.1^n)}{1 - 1.1}$$

$$\Rightarrow \quad -\frac{100}{10} = 1 - 1.1^n$$

$$\Rightarrow \quad 1.1^n = 11$$

$$\Rightarrow \quad n = \frac{\log_{10} 11}{\log_{10} 1.1} = 25.15885793$$

So Richard cycles for 26 days

On the 25th day Richard cycles $10 \times (1.1)^{24} = 98.497..$ miles and at the end of the 25th day he has covered

$$S_{25} = \frac{10(1 - (1.1)^{25})}{1 - 1.1} = 983.470.. \text{ miles}$$

So on the 26th day although he would be due to cycle 108.4 miles if he kept going, he can stop after 16.5 miles as he's reached his 1000 mile target.

Hence the furthest he cycles on any day is about 98.5 miles, on day 25.

9 A savings scheme is offering a rate of interest of 3.5% per annum for the lifetime of the plan. Alan wants to save up £20000. He works out that he can afford to save £500 every year which he will deposit on 1 January. If interest is paid on 31 December, how many years will it be before he has saved up his £20000

at end of 1st year his £500 investment is worth 500×1.035

$u_1 = 517.5$

at end of 2nd year this £500 is worth

$u_2 = 535.61$

and $u_3 = 554.36$

So as with Jane's investment in question 4 above the most recent sum invested makes the first term in the series and the oldest sum invested is the last term in the series.

The total investment pot has value S_n after n years where

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{517.5(1-1.035^n)}{1-1.035}$$

We want

$$\frac{517.5(1-1.035^n)}{-0.035} > 20000$$

$$\Rightarrow \quad 1 - 1.035^n < \frac{20000 \times -0.035}{517.5}$$

$$\Rightarrow \quad 1.035^n > 1 + \frac{280}{207}$$

$$\Rightarrow \quad n > \frac{\log_{10}\left(\frac{487}{207}\right)}{\log_{10}\left(\frac{207}{200}\right)} = 24.86947233$$

hence Alan needs to save for 25 years.